A Fourier–Borel transform for monogenic functionals

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Abstract

We discuss the Fourier-Borel transform for the dual of spaces of monogenic functions. This transform may be seen as a restriction of the classical Fourier-Borel transform for holomorphic functionals, and it transforms spaces of monogenic functionals into quotients of spaces of entire holomorphic functions of exponential type. We prove that, for the Lie ball, these quotient spaces are isomorphic to spaces of monogenic functions of exponential type.

Key words: Fourier-Borel transform, Lie ball, analytic functionals, monogenic functionals.
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1 Introduction and preliminaries

The classical Fourier-Borel transform is an extension of the Fourier transform to the spaces of analytic or holomorphic functionals (the duals of the spaces of analytic functions or holomorphic functions). It transforms analytic functionals into entire holomorphic functions of exponential type. A main result by A. Martineau states that the convex carrier of a functional can be determined by the exponential estimates of the Fourier-Borel transform. Fourier transforms for the spaces of hyperfunctions, and hence also for other function spaces which are subspaces of hyperfunctions, can be seen as a restriction of the Fourier-Borel transform.

In this paper, we focus on the dual of the spaces of left (or right) monogenic functions, which are defined as solutions of the left (or right) Dirac or of the generalized Cauchy-Riemann equation. These dual spaces, of so-called monogenic functionals, admit Hahn-Banach extensions belonging to spaces of analytic or holomorphic functionals, so that the Fourier-Borel transform in the sense of A. Martineau, becomes available for spaces of monogenic functionals.

The main problem is to determine the Fourier-Borel image of the functionals vanishing on spaces of monogenic functions. This restricted Fourier-Borel transform maps spaces of monogenic functionals into quotient spaces of entire holomorphic functions of exponential type. These quotient spaces may, in some cases, be interpreted as spaces of complex monogenic functions of exponential type, and so the Fourier-Borel transform may be re-interpreted as a transform which maps monogenic functionals into monogenic functions.

The development of our theory goes in several stages. In Section 2 we recall the duality theory for spaces of monogenic functions as established in the paper by R. Delanghe and F. Brackx [2]. This includes the Cauchy transform of a monogenic functional which leads to an isomorphism from spaces of monogenic functionals on a compact set K into spaces of functions monogenic outside the compact set K and vanishing at infinity. We also discuss the embedding of monogenic functionals into spaces of analytic or holomorphic functionals and, in particular, into the space of holomorphic functionals on the Lie ball. The Lie ball is important because every monogenic function on the unit ball of $\mathbb{R}^{m+1}$ admits a holomorphic extension to the Lie ball, a result that has been proved for harmonic functions by, for instance, J. Siciak [13].

In the sequel, we will also make substantial use of the theory of spherical harmonics and spherical monogenics on the Lie sphere, as it has been established in the papers by M. Morimoto [10] and F. Sommen [17].

In Section 3, we start by recalling the Fourier-Borel transform for spaces of holomorphic functionals and a main result by A. Martineau, (see Theorem 3.3). Next, we adapt this theorem to the situation of monogenic functionals. This leads to a Fourier-Borel transform with values in a quotient space of entire holomorphic functions of exponential type. In a first subsection, we discuss the Fischer decomposition and its dual for spaces of holomorphic functions and functionals on the Lie ball; see, in particular, Theorem 3.8. In a second subsection, we apply these results to the Fourier-Borel transform for monogenic functionals in the Lie ball, see Theorem 3.12. This Fourier-Borel transform takes values in the space of left monogenic functions of exponential type.

In Section 4 we revise the Fischer decomposition of the Fourier-Borel kernel whose expression is given in terms of the Gegenbauer polynomials.

In the fifth and last section we discuss transforms which are related to the Fourier-Borel transform. First of all, we show that the Fourier-Borel transform can be re-interpreted as a transform mapping functionals into holomorphic functions on the nullcone. We also present a Gabor-Fourier-Borel transform satisfying the heat equation and, finally, we discuss a related integral transform that may be expressed in terms of Bessel functions.

We now present some preliminary definitions and notations. Let us denote by $\mathbb{R}_m$ the real Clifford algebra over m imaginary units $\underline{e}_1, \dots, \underline{e}_m$ satisfying the relations $\underline{e}_i\underline{e}_j + \underline{e}_j\underline{e}_i = -2\delta_{ij}$. As customary, an element x in the Clifford algebra will be denoted by $x = \sum_A \underline{e}_A x_A$ where $x_A \in \mathbb{R}$, $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi–index, $\underline{e}_A = \underline{e}_{i_1}\underline{e}_{i_2} \dots \underline{e}_{i_r}$ and $\underline{e}_\emptyset = 1$.

In the Clifford algebra $\mathbb{R}_m$, we can identify the so called 1-vectors, namely the linear combinations with real coefficients of the elements $\underline{e}_i$, $i = 1, \dots, m$, with the vectors in the Euclidean space $\mathbb{R}^m$. The correspondence is given by the map the map $(x_1, x_2, \dots, x_m) \mapsto \underline{x} = x_1\underline{e}_1 + \dots + x_m\underline{e}_m$.

Analogously, an element $(x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$ will be identified with the element $x = x_0 + \underline{x}$ which is called paravector. The product of two 1-vectors splits into a scalar and a vector part:

$$\underline{x}\,\underline{u} = -\langle \underline{x}, \underline{u} \rangle + \underline{x} \wedge \underline{u}$$

where $\langle \underline{x}, \underline{u} \rangle = -\frac{1}{2}(\underline{x}\,\underline{u} + \underline{u}\,\underline{x})$ and $\underline{x} \wedge \underline{u} = \frac{1}{2}(\underline{x}\,\underline{u} - \underline{u}\,\underline{x})$.

The norm of a paravector x is defined as $|x|^2 = x_0^2 + x_1^2 + \dots + x_m^2$. The real part x_0 of x will be also denoted by $\mathrm{Re}(x)$. A function $f : U \subseteq \mathbb{R}^{m+1} \to \mathbb{R}_m$ is seen as a function $f(x)$ of the paravector x.

In the sequel, we will also consider the complexified Clifford algebra $\mathbb{C}_m$ which is generated over $\mathbb{C}$ by the imaginary units $\underline{e}_i$, $i = 1, \dots, m$. An element $(z_1, \dots, z_m) \in \mathbb{C}^m$ will be identified with the element $\underline{z} = \sum_{j=1}^m \underline{e}_j z_j \in \mathbb{C}_m$, $z_j = x_j + iy_j$, $j = 1, \dots, m$.

2 Monogenic functionals

The class of functions we will consider in this paper are in the kernel of the so-called generalized Cauchy-Riemann operator which is defined as

$$\partial_{x_0} + \partial_{\underline{x}} \qquad \text{where } \partial_{\underline{x}} = \sum_{i=1}^{m} \underline{e}_i \partial_{x_i}.$$

Definition 2.1. *Let Ω be an open set in $\mathbb{R}^{m+1}$. A real differentiable function $f : \Omega \to \mathbb{R}_m$ is said to be (left) monogenic if*

$$(\partial_{x_0} + \partial_{\underline{x}}) f(x_0, \underline{x}) = 0,$$

while f is said to be right monogenic if

$$f(x_0, \underline{x})(\partial_{x_0} + \partial_{\underline{x}}) = 0,$$

The set of functions (left) monogenic (resp. right monogenic) on Ω is denoted by $\mathcal{M}_\ell(\Omega)$ (resp. $\mathcal{M}_r(\Omega)$).

The set $\mathcal{M}_\ell(\Omega)$ is a right $\mathbb{R}_m$-module, while $\mathcal{M}_r(\Omega)$ is a left $\mathbb{R}_m$-module. If $K \subset \mathbb{R}^{m+1}$ is a compact set, we set

$$\mathcal{M}_\ell(K) = \text{ind} \lim_{\Omega \text{ open} \supset K} \mathcal{M}_\ell(\Omega) \qquad \mathcal{M}_r(K) = \text{ind} \lim_{\Omega \text{ open} \supset K} \mathcal{M}_r(\Omega).$$

In the sequel we will mainly consider $\mathcal{M}_r(K)$. As it is well known, see [15], $\mathcal{M}_r(K)$ is equipped with the inductive limit topology which is an LF-topology since the sets $\mathcal{M}_r(\Omega)$ are Fréchet modules, see [3].

Definition 2.2. *The set of right $\mathbb{R}_m$-linear functionals $T : \mathcal{M}_r(\mathbb{R}^{m+1}) \to \mathbb{R}_m$ is denoted by $\mathcal{M}'_r(\mathbb{R}^{m+1})$ and we call an element $T \in \mathcal{M}'_r(\mathbb{R}^{m+1})$ a monogenic functional.*
The set of $\mathbb{R}_m$-linear functionals $T : \mathcal{M}_r(K) \to \mathbb{R}_m$ is denoted by $\mathcal{M}'_r(K)$ and we call an element $T \in \mathcal{M}'_r(K)$ a monogenic functional on K.

Remark 2.3. It is immediate that the set of monogenic functionals is a left $\mathbb{R}_m$-module.

We also note that the action of a monogenic functional can be described as a formal integral

$$T[f] = \int f(x)T(x)dx.$$

Example 2.4. Let K be a compact set, $K \subset \Omega$ where Ω is an open set in $\mathbb{R}^{m+1}$.

1. Let $\varphi \in \mathcal{D}(K)$, the set of test functions supported by K. Then T_φ defined by

$$T_\varphi[f] := \int_{\mathbb{R}^{m+1}} f(x)\varphi(x)dx$$

is a monogenic functional.

2. Let $F \in \mathcal{E}'(K)$, the set of distributions supported by K. Then T_F defined by

$$T_F[f] := \int_{\mathbb{R}^{m+1}} f(x)F(x)dx$$

is a monogenic functional.
In particular, $\delta_u[f] = f(u) = \int_{\mathbb{R}^{m+1}} \delta(u-x)f(x)dx$.

In order to state next definition, we need to recall that the Cauchy kernel for monogenic functions is, see [3]:

$$E(x) = E(x_0 + \underline{x}) = \frac{1}{\omega_{m+1}} \frac{x_0 - \underline{x}}{|x_0 + \underline{x}|^{m+1}},$$

where ω_{m+1} is the surface area of the unit sphere S^m.

Definition 2.5. *The Cauchy transform of a functional T (also known as Fantappié indicatrix of the functional T) is defined by*

$$\widehat{T}(x) = \int E(x-u)T(u)du = T_u[E(x-u)].$$

If $T \in \mathcal{M}_r(K)'$ then $\widehat{T} \in \mathcal{M}^0_\ell(\mathbb{R}^{m+1} \setminus K)$ where $\mathcal{M}^0_\ell(\mathbb{R}^{m+1} \setminus K)$ denotes the $\mathbb{R}_m$-module of functions monogenic outside K which vanish at infinity. Also the converse is true, namely, any element in $\mathcal{M}^0_r(\mathbb{R}^{m+1} \setminus K)$ defines a functional in $\mathcal{M}'_r(K)$. In fact, let $f \in \mathcal{M}^0_\ell(\mathbb{R}^{m+1} \setminus K)$ and $g \in \mathcal{M}_r(K)$. Then there exists a function $\tilde{g}$ monogenic in an open set $\Omega \supset K$ which extends g. Consider a compact $K' \subset \Omega$ with smooth boundary and such that K is contained in the interior of K'. We define a functional $T_f \in \mathcal{M}_r(K)'$ by setting

$$T_f[g] = \int_{\partial K'} \tilde{g}(x) d\sigma_x f(x)$$

where

$$d\sigma_x = \sum_{j=0}^{m} (-1)^{j+1} \underline{e}_j \, dx_0 \wedge \widehat{dx_j} \wedge \ldots \wedge dx_m$$

and $\widehat{dx_j}$ means that dx_j is omitted. This construction does not depend on the choices of K' and $\tilde{g}$. This one-to-one correspondence has been proved in [2] and is precisely stated in the next result:

Theorem 2.6. *Let $K \subset \mathbb{R}^{m+1}$ be a compact set. Then*

$$\mathcal{M}'_\ell(K) \cong \mathcal{M}^0_r(\mathbb{R}^{m+1} \setminus K), \qquad \mathcal{M}'_r(K) \cong \mathcal{M}^0_\ell(\mathbb{R}^{m+1} \setminus K),$$

where $\mathcal{M}^0_r(\mathbb{R}^{m+1} \setminus K)$ (resp. $\mathcal{M}^0_\ell(\mathbb{R}^{m+1} \setminus K)$) denotes the set of right (resp. left) monogenic functions outside K which vanish at infinity.

Following [15], but see also [11], we now give the following definition:

Definition 2.7. *Let $\Omega \subseteq \mathbb{R}^m$ (or $\Omega \subseteq \mathbb{R}^{m+1}$). We denote by $\mathcal{A}_{(r)}(\Omega)$ the right $\mathbb{R}_m$-module of functions of the form $\sum_A f_A e_A$ where $f_A : \Omega \to \mathbb{C}$ are real analytic.*
We denote by $\mathcal{A}_{(\ell)}(\Omega)$ the left $\mathbb{R}_m$-module of functions of the form $\sum_A e_A f_A$ where $f_A : \Omega \to \mathbb{C}$ are real analytic.
By $\mathcal{A}'_{(r)}(\Omega)$ (resp. $\mathcal{A}'_{(\ell)}(\Omega)$) we denote the right (resp. left) $\mathbb{R}_m$-module of real analytic functionals on Ω with values in $\mathbb{R}_m$.

As we did before, by taking the inductive limit, for K compact set in $\mathbb{R}^m$ we define $\mathcal{A}'_{(r)}(K)$ and $\mathcal{A}'_{(\ell)}(K)$.

Remark 2.8. By the uniqueness of the Cauchy-Kowalevskaya extension, any real analytic function $f \in \mathcal{A}_{(\ell)}(K)$ can be extended to a unique function $\tilde{f} \in \mathcal{M}_r(K)$, where here K is thought as a subset of $\mathbb{R}^{m+1}$, see [14], Theorem 2.1. As a consequence

$$\mathcal{A}'_{(\ell)}(K) \cong \mathcal{M}^0_\ell(\mathbb{R}^{m+1} \setminus K).$$

To move from analytic functionals to monogenic functionals we note that if $K \subset \mathbb{R}^{m+1}$ then $\mathcal{M}_r(K) \subset \mathcal{A}_{(\ell)}(K)$ in fact every monogenic function is, in particular, real analytic (see [3]). Moreover, it is immediate that $\mathcal{M}_r(K)$ is a closed submodule of $\mathcal{A}_{(\ell)}(K)$.

Let us recall that for any hyperfunction T the hyperfunctions $\partial_{x_j}T$ and $\underline{e}_j T$ are well defined. In particular, if $T \in \mathcal{A}'_{(\ell)}(K)$ and $f \in \mathcal{A}_{(\ell)}(K)$ then $\partial_{x_j}T$ and $\underline{e}_j T$ act as follows

$$(\partial_{x_j}T)[f] = T[\partial_{x_j} f] \qquad \text{and} \qquad (\underline{e}_j T)[f] = T[f\underline{e}_j],$$

which is in accordance with the kernel representation

$$T[f] = \int f(x)T(x)dx.$$

The Cauchy transform can be defined also in the case of real analytic functionals:

Definition 2.9. *Let $T \in \mathcal{A}'_{(\ell)}(K)$, and let $E(x)$ be the Cauchy kernel of monogenic functions. The Cauchy transform $\widehat{T}$ of T is defined as*

$$\widehat{T}(x) := T_u * E = T_u[E(x-u)].$$

Note that in the previous definition the Cauchy kernel $E(x)$ is considered as a hyperfunction in the whole space.
We now prove:

Lemma 2.10. *Let $T \in \mathcal{A}'_{(\ell)}(K)$ be such that its restriction to $\mathcal{M}_r(K)$ is zero. Then $S = \widehat{T} \in \mathcal{A}'_{(\ell)}(K)$ or, in other words, we have $(\partial_{x_0} + \partial_{\underline{x}})S = T$ for some $S \in \mathcal{A}'_{(\ell)}(K)$.*

Proof. If the restriction of T to $\mathcal{M}_r(K)$ vanishes, then for $x = x_0 + \underline{x}$ outside K, $\widehat{T}(x) = T_u[E(x-u)] = 0$ which proves that the support of $\widehat{T}$ is contained inside K or $S = \widehat{T}$ belongs to $\mathcal{A}'_{(\ell)}(K)$. Moreover we also have that

$$(\partial_{x_0} + \partial_{\underline{x}})S = T * [(\partial_{x_0} + \partial_{\underline{x}})E] = T * \delta = T$$

which completes the proof. □

Theorem 2.11. *Let K be a compact set in $\mathbb{R}^{m+1}$. Then*

$$\mathcal{M}'_r(K) \cong \frac{\mathcal{A}'_{(\ell)}(K)}{(\partial_{x_0} + \partial_{\underline{x}})\mathcal{A}'_{(\ell)}(K)}.$$

Proof. Since $\mathcal{A}_{(\ell)}(K)$ is locally convex then the restriction map $\mathcal{A}'_{(\ell)}(K) \to \mathcal{M}'_r(K)$ is surjective by the Hahn-Banach theorem. Thus any functional $T \in \mathcal{M}'_r(K)$ extends to a functional $F \in \mathcal{A}'_{(\ell)}(K)$. Moreover, due to Lemma 2.10, if for a given $F \in \mathcal{A}'_{(\ell)}(K)$ the restriction T to $\mathcal{M}'_r(K)$ vanishes, then F has the form $F = (\partial_{x_0} + \partial_{\underline{x}})S$ for some S. This proves the result. □

Monogenic functionals, as well as analytic functionals, do not possess a well defined support, so the corresponding notion is the one of carrier, that we recall below:

Definition 2.12. *We say that a monogenic functional $T \in \mathcal{M}'_r(\mathbb{R}^{m+1})$ is carried by the compact set K (and K is said to be a carrier) if for every open neighborhood Ω of K there is a positive constant c_Ω such that for every $f \in \mathcal{M}_r(\mathbb{R}^{m+1})$ it is*

$$|T(f)| \leq c_\Omega \sup_{x\in\Omega} |f(x)|.$$

Note that a carrier of a functional is not unique, in fact if K is carrier, any other compact set containing K is a carrier. A carrier is not unique not even when we impose that it is minimal, namely that no proper closed subset of it is a carrier. For uniqueness more hypothesis on a minimal carrier are needed.
Assume that a functional $T \in \mathcal{M}'_r(\mathbb{R}^{m+1})$ is carried by a compact K and suppose that K has no holes. By Runge's theorem, see [3], $\mathcal{M}_r(\mathbb{R}^{m+1})$ is dense in $\mathcal{M}_r(K)$.
In the sequel we will assume that K has no holes, is minimal and convex.

Let us now extend the previous discussion and let us consider $\mathbb{C}^m$, whose elements will be denoted by $\underline{z} = \sum_{j=1}^m \underline{e}_j z_j$, $z_j = x_j + iy_j \in \mathbb{C}$, $j = 1, \dots, m$, and the complexified Clifford algebra $\mathbb{C}_m$. We can also write $\underline{z} = \underline{x} + i\underline{y}$, with obvious meaning of the symbols. Let Ω be an open subset of $\mathbb{C}^{m+1}$ or $\mathbb{C}^m$. The linear space $\mathcal{O}(K)$ of holomorphic functions on K is defined, as customary, as inductive limit. Similarly to what we have done before, we can define the $\mathbb{C}_m$-modules $\mathcal{O}_{(\ell)}(\Omega)$ as the set of elements of the form $\sum_A e_A F_A$ where $F_A \in \mathcal{O}(\Omega)$. In an analogous way, one defines $\mathcal{O}_{(r)}(\Omega)$ and, as before, $\mathcal{O}_{(\ell)}(K)$ and $\mathcal{O}_{(r)}(K)$ where K is a compact in $\mathbb{C}^{m+1}$ or $\mathbb{C}^m$. Let us set $\partial_z = \partial_{z_0} + \sum_{j=1}^m \underline{e}_j \partial_{z_j} = \partial_{z_0} + \partial_{\underline{z}}$.
Similar to Theorem 2.11 one may expect the following property:

Property 2.13. *Let K be a compact set in $\mathbb{C}^{m+1}$. Then*

$$\mathcal{M}'_r(K) \cong \frac{\mathcal{O}'_{(\ell)}(K)}{(\partial_{z_0} + \partial_{\underline{z}})\mathcal{O}'_{(\ell)}(K)}. \tag{1}$$

However, while the Hahn-Banach extension theorem is still available, the Lemma 2.10 does not generalize to compact subsets of $\mathbb{C}^{m+1}$. It still holds for $K \subset \mathbb{R}^{m+1} \subset \mathbb{C}^{m+1}$ and also for very special domains like the Lie ball.

Definition 2.14. *A compact set $K \subset \mathbb{C}^{m+1}$ is said to be admissible if the isomorphism* (1) *holds.*

Of course, the previous ideas and constructions can be easily adapted to the case of Dirac operators $\partial_{\underline{z}}$ in $\mathbb{C}^m$. From this point on, we will work in $\mathbb{C}^m$.

Example 2.15. The above Property 2.13 applies to the case in which K is the closure of the unit ball $\overline{B_{\mathbb{R}}(0,1)}$ in $\mathbb{R}^m$. A fundamental system of open, convex sets containing K is given by $\Omega_\varepsilon = B_{\mathbb{R}}(0,(1+\varepsilon)) + iB_{\mathbb{R}}(0,\varepsilon) \subset \mathbb{C}^m$ and $\mathcal{M}_r(K)$ can be obtained as an inductive limit of such open sets Ω_ε. It appears that this choice of the compact K does not lead to a Fischer decomposition for the space $\mathcal{O}_{(\ell)}(K)$. The only example we know of an admissible compact set where the Fischer decomposition holds for the space $\mathcal{O}_{(\ell)}(K)$ is the case in which K is the closure of the Lie ball.

Definition 2.16. *The Lie norm $L(\underline{z})$ is given by*

$$L^2(\underline{z}) := |\underline{x}|^2 + |\underline{y}|^2 + 2|\underline{x} \wedge \underline{y}|,$$

while the Lie ball is defined as

$$LB(0,1) = \{\underline{z} \in \mathbb{C}^m \;:\; L(\underline{z}) < 1\}.$$

It is immediate to verify that $L(\underline{z}) \geq |\underline{z}|$ hence it is clear that if $f(\underline{x}) \in \mathcal{M}_\ell(\overline{B(0,1)})$ then $f(\underline{z}) \in \mathcal{M}_\ell(\overline{LB(0,1)})$, see also [13].
It is also true that if $f(\underline{x}) \in \mathcal{M}_r(\overline{B_{\mathbb{R}}(0,1)})$ then its complex extension $f(\underline{z}) \in \mathcal{M}_r(\overline{LB(0,1)})$ and the Lie ball is the maximal domain with such extension property (see also [13]).

3 The Fourier-Borel transform

Let us begin this section by briefly revising some basics on the Fourier-Borel transform in the complex case. The Fourier-Borel transform of $T \in \mathcal{O}'(\mathbb{C})$ is defined as

$$\mathcal{F}T(z) := T_u[e^{uz}].$$

The functional T is carried by $\overline{B(0,R)}$ if and only if $|\mathcal{F}T(z)| \leq C_\varepsilon e^{(R+\varepsilon)|z|}$, i.e. $\mathcal{F}T$ is of exponential type. More in general, we have:

Definition 3.1. *Given $T \in \mathcal{O}'(\mathbb{C}^m)$ its Fourier-Borel transform is defined by*

$$\mathcal{F}T(\underline{z}) := T_{\underline{u}}[e^{\langle \underline{u}, \underline{z}\rangle}],$$

where $\langle \underline{u}, \underline{z}\rangle = \sum_{j=1}^m u_j z_j$, $\underline{u} = \sum_{j=1}^m \underline{e}_j u_j$, $\underline{z} = \sum_{j=1}^m \underline{e}_j z_j$, $u_j, z_j \in \mathbb{C}$.

A well known fundamental result, see [9], is the following

Theorem 3.2. *A functional $T \in \mathcal{O}'(\mathbb{C}^m)$ is carried by $\overline{B(0,R)}$ if and only if $|\mathcal{F}T(\underline{z})| \leq C_\varepsilon e^{(R+\varepsilon)|\underline{z}|}$.*

Let K be a compact convex set in $\mathbb{C}^m$. To assume convexity is not reductive, since we can always take the convex hull of K.
We define the so-called supporting function of a compact, convex K as:

$$H_K(\underline{z}) = \sup_{\underline{u} \in K} \mathrm{Re}\langle \underline{u}, \underline{z}\rangle$$

(note that H_K is a polar norm of $\underline{z}$). It follows from Theorem 3.2 that, see [9]:

Theorem 3.3. *The compact set K is a convex carrier of $T \in \mathcal{O}'(\mathbb{C}^m)$ if and only if*

$$|\mathcal{F}T(\underline{z})| \leq C_\varepsilon \exp(H_K(\underline{z}) + \varepsilon|\underline{z}|),$$

where C_ε is a suitable positive number.

In the case $K = \overline{LB(0,1)}$, we denote the supporting function $H_K(\underline{z})$ by $L^\triangle(\underline{z})$.
Let us now consider the Clifford analysis setting. Let K be an admissible compact, convex set in $\mathbb{C}^m$ and let $T \in \mathcal{M}'_r(K)$. In view of Definition 2.14 we can associate to T an element $T + \partial_{\underline{u}} S$ where $S \in \mathcal{O}'_{(\ell)}(K)$. Thus we define $\mathcal{F}T$ as

$$T_{\underline{u}}[\exp\langle \underline{u}, \underline{z}\rangle] + \partial_{\underline{u}} S[\exp\langle \underline{u}, \underline{z}\rangle] = \mathcal{F}T + \mathcal{F}(\partial_{\underline{u}} S)$$

where

$$\mathcal{F}(\partial_{\underline{u}} S) = \int \exp\langle \underline{u}, \underline{z}\rangle \partial_{\underline{u}} S(\underline{u})\, d\underline{u} = -\underline{z}\mathcal{F}S(\underline{z}).$$

Let us consider the following space of Clifford algebra-valued entire functions with growth conditions:

$$\mathrm{Exp}_K = \{f \in \mathcal{O}_{(\ell)}(\mathbb{C}^m) \ : \ |f(\underline{z})| \leq C_\varepsilon \exp(H_K(\underline{z}) + \varepsilon|\underline{z}|)\}.$$

We will say that a function f belonging to Exp_K is of exponential type.
Theorem 3.3 can be rephrased as follows, see [9]:

Theorem 3.4. *Let K be a compact convex set. The Fourier-Borel transform provides the isomorphism*

$$\mathcal{O}'_{(\ell)}(K) \cong \mathrm{Exp}_K.$$

We also have the following:

Theorem 3.5. *Let K be an admissible compact convex set. Then the Fourier-Borel transform*

$$\mathcal{F} : \ \mathcal{M}'_r(K) \to \mathrm{Exp}_K / \underline{z}\, \mathrm{Exp}_K$$

is an isomorphism.

Proof. Due to the isomorphism (1), every $T \in \mathcal{M}'_r(K)$ may be identified with an equivalence class of holomorphic functionals of the form $F + \partial_{\underline{z}} S$ with $F, S \in \mathcal{O}'_{(\ell)}(K)$. Hence the Fourier-Borel transform is in fact given by $\mathcal{F}(F) + \mathcal{F}(\partial_{\underline{z}} S)$ which is equal to $\mathcal{F}(F) - \underline{z}\mathcal{F}(S)$ where both $\mathcal{F}(F)$ and $\mathcal{F}(S)$ are of exponential type on K. This proves the result. □

Example 3.6. Examples of convex sets for which the above results make sense are $K = \overline{B(0,1)} \cap \mathbb{R}^m$, $K = \overline{LB(0,1)}$. The main problem is to investigate when the quotient $\mathrm{Exp}_K/\underline{z}\mathrm{Exp}_K$ corresponds to a space of entire monogenic functions that still satisfies the same exponential estimates. It turns out that this will be only possible in the second case, namely for $K = \overline{LB(0,1)}$.

3.1 Fischer decomposition and its dual

Let $f \in \mathcal{O}(\mathbb{C}^m)$. Then the decomposition

$$f(\underline{z}) = M(f)(\underline{z}) + \underline{z} g(\underline{z})$$

where $g \in \mathcal{O}(\mathbb{C}^m)$ and $\partial_{\underline{z}} M(f)(\underline{z}) = 0$ is called Fischer decomposition of f. There is also a Fischer decomposition for $f \in \mathcal{O}(\overline{LB(0,1)})$, see [17]:

$$f(\underline{z}) = \sum_{k,s} \underline{z}^s P_{k,s}(\underline{z}) \tag{2}$$

where $P_{k,s}(\underline{z})$ is a spherical monogenic of degree k which can be rewritten in the form

$$f(\underline{z}) = \sum_{s=0}^{\infty} \underline{z}^s M_s(f)(\underline{z})$$

where $\partial_{\underline{z}} M_s(f)(\underline{z}) = 0$. Note that $M_0(f) = M(f)$.
Using the decomposition (2) we can prove the following:

Lemma 3.7. *Let $f \in \mathcal{O}(\overline{LB(0,1)})$. Then there exists a unique $g \in \mathcal{O}(\overline{LB(0,1)})$ such that*

$$f(\underline{z}) = (g(\underline{z})\underline{z})\partial_{\underline{z}}.$$

Proof. In $\mathcal{O}(\overline{LB(0,1)})$ we can decompose f as

$$f(\underline{z}) = \sum_{k,s} P_{k,s}(\underline{z})\underline{z}^s$$

where $P_{k,s}(\underline{z})$ is a function right spherical monogenic of degree k. In general, we have

$$(P_{k,s}(\underline{z})\underline{z}^s)\partial_{\underline{z}} = \beta_{k,s} P_k(\underline{z})\underline{z}^{s-1}$$

where $\beta_{s,k}$ satisfy

$$\beta_{2s,k} = -2s, \qquad \beta_{2s+1,k} = -(2s+2k+m). \tag{3}$$

hence we deduce that $\beta_{s+1,k} \neq 0$ and $|\beta_{s+1,k}| \geq 1$. Let $g(\underline{z}) = \sum_{k,s} \frac{1}{\beta_{s+1,k}} P_{k,s}(\underline{z})\underline{z}^s$, then $g \in \mathcal{O}(\overline{LB(0,1)})$. A direct computation shows that $f(\underline{z}) = (g(\underline{z})\underline{z})\partial_{\underline{z}}$. □

Theorem 3.8. *Let $K = \overline{LB(0,1)}$ and let $T \in \mathcal{O}'_{(\ell)}(K)$. Then there exists a unique $S \in \mathcal{O}'_{(\ell)}(K)$ such that for all $g \in \mathcal{O}_{(\ell)}(K)$ one has*

$$(T - \partial_{\underline{z}} S)[g(\underline{z})\underline{z}] = 0.$$

Proof. Let $T \in \mathcal{O}'_{(\ell)}(K)$, we have to find $S \in \mathcal{O}'_{(\ell)}(K)$ with the above property. This amounts to determine $S[f]$ for every $f \in \mathcal{O}_{(\ell)}(K)$. So let $f \in \mathcal{O}_{(\ell)}(K)$ and let $g \in \mathcal{O}_{(\ell)}(K)$ be the unique solution of $f(\underline{z}) = -(g(\underline{z})\underline{z})\partial_{\underline{z}}$ which exists by Lemma 3.7. Now, by the above assumption S has to be the solution of

$$(T - \partial_{\underline{z}} S)[g(\underline{z})\underline{z}] = 0,$$

which, with the above choice of g leads to

$$T[g(\underline{z})\underline{z}] = (\partial_{\underline{z}} S)[g(\underline{z})\underline{z}] = -S[(g(\underline{z})\underline{z})\partial_{\underline{z}}] = S[f].$$

This uniquely determines $S \in \mathcal{O}'_{(\ell)}(K)$. □

A function $f \in \mathrm{Exp}_K$ decomposes, as an element in $\mathcal{O}(\mathbb{C}^m)$, as $f = M(f) + \underline{z}g$. In the next subsection we will show that both $M(f)$ and g belong to Exp_K when K is the closure of the Lie ball $LB(0,1)$. To show this result we make use the following:

Corollary 3.9. *(Dual Fischer decomposition) Let $T \in \mathcal{O}'_{(\ell)}(K)$, then T admits a unique decomposition of the form*

$$T = M(T) + \partial_{\underline{z}} S$$

where $M(T)$ vanishes on all functions of the form $g(\underline{z})\underline{z}$.

$M(T)$ will be called the *monogenic part* of the functional T.
It is clear that for every $f \in \mathcal{M}_r(\overline{LB(0,1)})$

$$T[f] = M(T)[f].$$

It is also clear that when $f \in \mathcal{O}_{(\ell)}(\overline{LB(0,1)})$ and $f = M(f) + g(\underline{z})\underline{z}$ denotes the Fischer decomposition, then

$$M(T)[f] = M(T)[M(f)].$$

Definition 3.10. *Let $\Omega \subseteq \mathbb{C}^m$. We denote by $\mathcal{M}_{\ell,k}(\Omega)$ (resp. $\mathcal{M}_{r,k}(\Omega)$) the set of solutions of the equation $\partial_{\underline{z}}^k f(\underline{z}) = 0$ (resp. $f(\underline{z})\partial_{\underline{z}}^k = 0$), called k-monogenic functions (resp. k-right monogenic functions).*

If $K \subset \mathbb{C}^m$ is a compact set, by taking the inductive limit we can define $\mathcal{M}_{\ell,k}(K)$ (resp. $\mathcal{M}_{r,k}(K)$). In the next remark, K is the closed Lie ball:

Remark 3.11. The Almansi decomposition theorem states that $f(\underline{z}) \in \mathcal{M}_{\ell,k}(\overline{LB(0,1)})$ may be decomposed as

$$f(\underline{z}) = f_0(\underline{z}) + \underline{z} f_1(\underline{z}) + \ldots + \underline{z}^{k-1} f_{k-1}(\underline{z})$$

where $f_0, \ldots, f_{k-1}$ are left monogenic on the closed Lie ball. The set of k-monogenic functionals contains elements T belonging to $\mathcal{M}'_{r,k}(\overline{LB(0,1)})$ where $\mathcal{M}_{r,k}(\overline{LB(0,1)})$ is the set of right k-monogenic functions on $\overline{LB(0,1)}$. Such functionals admit a dual Almansi decomposition of the form

$$T = T_0 + \partial_{\underline{z}} T_1 + \ldots + \partial_{\underline{z}}^{k-1} T_{k-1},$$

where $T_j[g(\underline{u})\underline{u}] = 0$ i.e. $T_j \in \mathcal{M}'_r(\overline{LB(0,1)})$.

3.2 Fourier-Borel transform for monogenic functionals in the Lie ball

Assume that $f(\underline{z})$ is a Clifford algebra valued entire function satisfying the estimate

$$|f(\underline{z})| \leq C_\varepsilon \exp((1+\varepsilon)L^{\triangle}(\underline{z})).$$

Then there exists a holomorphic functional $F \in \mathcal{O}'_{(\ell)}(\overline{LB(0,1)})$ such that $\mathcal{F}F = f$, see [9] and Theorem 3.4. This is crucial to prove next result on monogenic functionals in the Lie ball.

Theorem 3.12. *Let $T \in \mathcal{M}'_r(\overline{LB(0,1)})$ and let $F \in \mathcal{O}'_{(\ell)}(\overline{LB(0,1)})$ be a Hahn-Banach extension of T. Then $\mathcal{F}T$ may be defined as*

$$\mathcal{F}T(\underline{z}) = \mathcal{F}M(F)(\underline{z})$$

and it is an entire left monogenic function belonging to $Exp_{\overline{LB(0,1)}}$.

Proof. Let $F \in \mathcal{O}'_{(\ell)}(\overline{LB(0,1)})$ be a Hahn-Banach extension of $T \in \mathcal{M}'_r(\overline{LB(0,1)})$. Consider the dual Fischer decomposition $F = M(F) + \partial_{\underline{u}} S$, see Corollary 3.9, then

$$\mathcal{F}M(F) = M(F)_{\underline{u}}(\exp\langle \underline{u}, \underline{z}\rangle) =: g(\underline{z}).$$

The function $g(\underline{z})$ satisfies the same exponential estimate as the function f such that $\mathcal{F}F = f$, namely

$$|g(\underline{z})| \leq C_\varepsilon \exp((1+\varepsilon)L^{\triangle}(\underline{z})).$$

Moreover, since

$$g(\underline{z}) = M(F)[M(\exp\langle \underline{u}, \underline{z}\rangle)]$$

and

$$\partial_{\underline{z}} M(\exp\langle \underline{u}, \underline{z}\rangle) = M(\exp\langle \underline{u}, \underline{z}\rangle)\partial_{\underline{u}} = 0$$

we have $\partial_{\underline{z}} g(\underline{z}) = 0$. So the relation $F = M(F) + \partial_{\underline{u}} S$ transforms to

$$f = g + \mathcal{F}(\partial_{\underline{u}} S) = g + (\partial_{\underline{u}} S)[\exp(\langle \underline{u}, \underline{z}\rangle)] = g(\underline{z}) + \underline{z} h(\underline{z})$$

where $g \in \mathrm{Exp}_K$ is left monogenic and $h(\underline{z}) = \mathcal{F}S(\underline{z}) \in \mathrm{Exp}_K$. $\square$

Remark 3.13. From now on we write the Fischer decomposition of the Fourier-Borel kernel as

$$\exp\langle \underline{u}, \underline{z}\rangle = E(\underline{z}, \underline{u}) + \underline{z}h(\underline{z}, \underline{u})\underline{u}$$

with

$$\partial_{\underline{z}} E(\underline{z}, \underline{u}) = E(\underline{z}, \underline{u})\partial_{\underline{u}} = 0.$$

Hence

$$\mathcal{F}M(F)(\underline{z}) = M(F)[E(\underline{z}, \underline{u})] = T_{\underline{u}}[E(\underline{z}, \underline{u})].$$

So the Fourier-Borel transform for monogenic functionals may in fact be defined as the mapping

$$\mathcal{F}:\ T \mapsto T_{\underline{u}}[E(\underline{z}, \underline{u})];$$

moreover, it is a topological isomorphism

$$\mathcal{M}'_r(\overline{LB(0,1)}) \cong Exp_{\overline{LB(0,1)}} \cap \mathcal{M}_\ell(\mathbb{C}^m).$$

4 Fischer decomposition of $\exp\langle \underline{u}, \underline{z}\rangle$

The study of the Fischer decomposition of the kernel of the Fourier-Borel transform $\exp(z_1w_1 + \ldots + z_mw_m)$, $z_i, w_i \in \mathbb{C}$, in the multi-dimensional case goes back to the work of H. S. Shapiro for $m = 3$, see [12], and has been generalized by J. Aniansson in [1] to higher dimensions. In the paper [6] the Fischer decomposition of the kernel $\exp\langle \underline{u}, \underline{z}\rangle$ has been studied in the framework of Clifford analysis and we briefly summarize below some results from [6].
First of all, we note that the Fischer decomposition gives the expansion

$$\frac{1}{k!}\langle \underline{x}, \underline{u}\rangle^k = \sum_{s=0}^{k} \underline{x}^s Z_{k,s}(\underline{x}, \underline{u})\underline{u}^s \tag{4}$$

where $Z_{k,s}$ are zonal monogenic functions of degree $k-s$. More specifically, $Z_{k,s}$ are homogeneous of degree $k-s$ in both $\underline{u}$ and $\underline{x}$, they are left monogenic in $\underline{x}$ and right monogenic in $\underline{u}$ and they have the form

$$Z_{k,s}(\underline{x}, \underline{u}) = \frac{1}{\beta_{s,k-s}\ldots\beta_{1,k-s}} Z_{k-s}(\underline{x}, \underline{u})$$

with $\beta_{2s,k}$, $\beta_{2s+1,k}$ as in (3).

$$Z_k(\underline{x}, \underline{u}) = \frac{\Gamma\left(\frac{m}{2}-1\right)}{2^{k+1}\Gamma\left(k+\frac{m}{2}\right)}(|\underline{x}|\,|\underline{u}|)^k\left[(k+m-2)C_k^{\frac{m}{2}-1}(t) + (m-2)\frac{\underline{x}\wedge\underline{u}}{|\underline{x}|\,|\underline{u}|}C_{k-1}^{\frac{m}{2}}(t)\right],$$

$t = \dfrac{\langle \underline{x}, \underline{u}\rangle}{|\underline{x}|\,|\underline{u}|}$ and C_k^r are the Gegenbauer polynomials. Thus, using (4), the Fischer decomposition of the kernel $\exp\langle \underline{u}, \underline{z}\rangle$ can be written as

$$\begin{aligned}
\exp\langle \underline{u}, \underline{z}\rangle &= \sum_{k=0}^{\infty}\frac{1}{k!}\langle \underline{x}, \underline{u}\rangle^k = \sum_{k=0}^{\infty}\sum_{s=0}^{k} \underline{x}^s Z_{k,s}(\underline{x}, \underline{u})\underline{u}^s \\
&= \sum_{s=0}^{\infty}\sum_{k=s}^{\infty} \underline{x}^s Z_{k,s}(\underline{x}, \underline{u})\underline{u}^s \\
&= \sum_{s=0}^{\infty} \underline{x}^s E_s(\underline{x}, \underline{u})\underline{u}^s
\end{aligned}$$

where we have set $E_s(\underline{x}, \underline{u}) = \sum_{k=s}^{\infty} Z_{k,s}(\underline{x}, \underline{u})$. Let us set

$$E(\underline{x}, \underline{u}) = E_0(\underline{x}, \underline{u}) = \sum_{k=0}^{\infty} Z_k(\underline{x}, \underline{u}). \tag{5}$$

To find a closed formula for the Fourier-Borel kernel is complicated. The functions E, E_s may be expressed in terms of Bessel functions as shown in [6].

5 Related transforms

In this section we present some transforms that are related to the Fourier-Borel transform. Specifically, we define the Fourier-Borel transform on the nullcone, a transform that involves solutions of the heat equation and a transform that can be expressed in terms of the Bessel functions.

5.1 The Fourier-Borel transform on the nullcone

Let us denote by $\mathcal{N}$ the nullcone in $\mathbb{C}^m$ i.e. the set of $\underline{z} \in \mathbb{C}^m$ such that $\underline{z}^2 = 0$. Then for $f \in \mathcal{O}_{(r)}(\mathbb{C}^m)$ with Fischer decomposition $f(\underline{z}) = M(f)(\underline{z}) + \underline{z}g(\underline{z})$ the restriction to the nullcone $\mathcal{N}$ is such that

$$\underline{z}f(\underline{z})_{|\mathcal{N}} = \underline{z}M(f)(\underline{z})_{|\mathcal{N}}.$$

Lemma 5.1. *Let $\underline{\tau} \in \mathcal{N}$ and consider the correspondence $\rho :\ \mathcal{O}_{(r)}(\mathbb{C}^m) \to \mathcal{O}_{(r)}(\mathcal{N})$ given by*

$$\rho(g) = \underline{\tau}g(\underline{\tau}).$$

Then ρ is injective on the closed subspace $\mathcal{M}_\ell(\mathbb{C}^m)$.

Proof. The condition $\underline{\tau}g(\underline{\tau}) = 0$ means that $\underline{z}g(\underline{z}) = \underline{z}^2h(\underline{z})$ for some $h \in \mathcal{O}_{(r)}(\mathbb{C}^m)$. This implies that $g(\underline{z}) = \underline{z}h(\underline{z})$, and assuming that g is also monogenic then it implies $g = 0$. $\square$

Let $F \in \mathcal{O}'_{(\ell)}(\mathbb{C}^m)$ then the restriction of $\underline{z}\mathcal{F}F(\underline{z}) = F_{\underline{u}}[\underline{z}\exp\langle \underline{u}, \underline{z}\rangle]$ to the nullcone is given by

$$\underline{\tau}\mathcal{F}F(\underline{\tau}) = \underline{\tau}\mathcal{F}M(F)(\underline{\tau}).$$

Hence for $T \in \mathcal{M}'_r(\mathbb{C}^m)$ we may find an Hahn-Banach extension $F \in \mathcal{O}'_{(\ell)}(\mathbb{C}^m)$ and for the Fourier-Borel transform of T given by

$$\mathcal{F}T(\underline{z}) = T_{\underline{u}}[E(\underline{u}, \underline{z})]$$

we have that

$$\underline{z}\mathcal{F}T(\underline{z}) = \underline{z}\mathcal{F}M(F)(\underline{z}).$$

This last relation restricts to the nullcone as

$$\underline{\tau}\mathcal{F}T(\underline{\tau}) = M(F)_{\underline{u}}[\underline{\tau}\exp\langle \underline{u}, \underline{\tau}\rangle] = F_{\underline{u}}[\underline{\tau}\exp\langle \underline{u}, \underline{\tau}\rangle],$$

since the map $\underline{u} \mapsto \underline{\tau}\exp\langle \underline{u}, \underline{\tau}\rangle$ is already monogenic in $\underline{u}$. Then we also have that

$$F_{\underline{u}}[\underline{\tau}\exp\langle \underline{u}, \underline{\tau}\rangle] = T_{\underline{u}}[\underline{\tau}\exp\langle \underline{u}, \underline{\tau}\rangle].$$

In $\mathbb{R}^{m+1}$ one may consider the transform

$$\mathcal{T}(F_{x_0,\underline{x}})(\underline{t}) = F_{x_0,\underline{x}}\left[\left(1 - \frac{i\underline{t}}{|\underline{t}|}\right)\exp\langle \underline{x}, \underline{t}\rangle - ix_0|\underline{t}|\right],$$

which is related to the Fourier-Borel transform on the nullcone and which makes use of the monogenic exponential function used in papers by K.I. Kou, C. Li, T. Qian and A. McIntosh, see [7] and [8].

5.2 The Gabor-Fourier-Borel transform

Consider the function $\exp(\langle \underline{x}, \underline{z}\rangle - t\underline{z}^2)$ where $(\underline{x}, t) \in \mathbb{R}^{m+1}$ and $\underline{z} \in \mathbb{C}^m$. Then this function satisfies the heat equation $(\Delta_{\underline{x}} - \partial_t)f(\underline{x}, t) = 0$.
Next, consider the space $\mathcal{M}_{r,s}(\mathbb{C}^m)$ of right s-monogenic functions in $\mathbb{C}^m$. Then we introduce the following:

Definition 5.2. *Let $F \in \mathcal{M}'_{r,s}(\mathbb{C}^m)$. Then the Gabor-Fourier-Borel transform of F is given by*

$$\mathcal{G}(F)(\underline{x}, t) = F_{\underline{z}}[\exp(\langle \underline{x}, \underline{z}\rangle - t\underline{z}^2)]$$

where F is represented by a functional $F \in \mathcal{O}_{(\ell)}(\mathbb{C}^m)$ satisfying

$$F[g(\underline{z})\underline{z}^s] = 0, \qquad g \in \mathcal{O}(\mathbb{C}^m).$$

We note that the Gabor-Fourier-Borel transform can be related with the Bargmann transform, though it is not the same.
Now consider the Fischer decomposition of the Gabor-Fourier-Borel kernel

$$\exp(\langle \underline{x}, \underline{z}\rangle - t\underline{z}^2) = \exp(-t\underline{z}^2)\sum_{k=0}^{\infty} \underline{x}^k E_k(\underline{x}, \underline{z})\underline{z}^k,$$

where the functions $E_k(\underline{x}, \underline{z})$ are defined in (5), which may be rewritten in the form

$$\sum_{k=0}^{\infty} \mathcal{G}_k(\underline{x}, t; \underline{z})\underline{z}^k$$

where $\mathcal{G}_k(\underline{x}, t; \underline{z})$ satisfies the heat equation $(\Delta_{\underline{x}} - \partial_t)f(\underline{x}, t; \underline{z}) = 0$ together with the monogenicity conditions $\partial_{\underline{x}}^{k+1} f(\underline{x}, t; \underline{z}) = f(\underline{x}, t; \underline{z})\partial_{\underline{z}} = 0$. Hence the Gabor-Fourier-Borel transform of F is given by

$$\mathcal{G}(F)(\underline{x}, t) = F_{\underline{z}}[\sum_{k=0}^{s-1} \mathcal{G}_k(\underline{x}, t; \underline{z})\underline{z}^k]$$

which clearly satisfies the heat equation $(\Delta_{\underline{x}} - \partial_t)f(\underline{x}, t) = 0$ together with the monogenicity conditions $\partial_{\underline{x}}^s f(\underline{x}, t) = 0$. It is a polynomial solution to the heat equation with respect to the variable t.
More in general, for $F \in \mathcal{O}'_{(\ell)}(\mathbb{C}^m)$ one may define a Gabor-Fourier-Borel transform by

$$\mathcal{G}(F)(\underline{x}, t) = F_{\underline{z}}[\exp(\langle \underline{x}, \underline{z}\rangle - t\underline{z}^2)] = \sum_{k=0}^{\infty} F_{\underline{z}}[\mathcal{G}_k(\underline{x}, t; \underline{z})\underline{z}^k],$$

which satisfies the heat equation, where the above series is an expansion in terms of Clifford algebra valued heat polynomials (namely, polynomials satisfying the heat equation).
Using the dual Fischer decomposition

$$F = \sum_{s=0}^{\infty} \partial_{\underline{z}}^s F_s, \qquad F_s \in \mathcal{M}_r'(\mathbb{C}^m)$$

the above transform may be written as

$$\mathcal{G}(F)(\underline{x}, t) = \sum_{k,s=0}^{\infty} (\partial_{\underline{z}}^s F_s)[\mathcal{G}_k(\underline{x}, t; \underline{z})\underline{z}^k]$$

$$= \sum_{k=0}^{\infty} F_k[(-1)^k \mathcal{G}_k(\underline{x}, t; \underline{z})\underline{z}^k \partial_{\underline{z}}^k].$$

5.3 The Bessel transform

Consider the function

$$B(\underline{x}, \underline{u}) = \exp\langle \underline{x}, \underline{u}\rangle J(\underline{x} \wedge \underline{u})$$

where

$$J(\underline{x} \wedge \underline{u}) = 2^{(m-3)/2}\Gamma\left(\frac{m-1}{2}\right) |\underline{x} \wedge \underline{u}|^{(3-m)/2} \left(J_{(m-3)/2}(|\underline{x} \wedge \underline{u}|) + \frac{\underline{x} \wedge \underline{u}}{|\underline{x} \wedge \underline{u}|} J_{(m-1)/2}(|\underline{x} \wedge \underline{u}|)\right),$$

and J_ℓ denotes the Bessel function of the first kind of order ℓ. Then the function $B(\underline{x}, \underline{u})$ satisfies the two sided monogenicity condition $\partial_{\underline{x}} B(\underline{x}, \underline{u}) = B(\underline{x}, \underline{u})\partial_{\underline{u}} = 0$. So one may define for $T \in \mathcal{M}_r'(\mathbb{C}^m)$ the so-called Bessel transform

$$\mathcal{B}T(\underline{z}) = T_{\underline{u}}[B(\underline{x}, \underline{u})]$$

which transforms monogenic functionals into left monogenic functions. This transform is not equivalent to the Fourier-Borel transform but, in [6], the Clifford-Bessel function $B(\underline{x}, \underline{u})$ has been related to the exponential function $E(\underline{x}, \underline{u})$ that appears in the Fourier-Borel transform. In fact, if one considers the restriction to parallel pairs of vectors $\underline{x} \parallel \underline{u}$ then we obtain that

$$B(\underline{x}, \underline{u})|_{\underline{x} \parallel \underline{u}} = \exp(|\underline{x}|\, |\underline{u}|)$$

while

$$E(\underline{x}, \underline{u})|_{\underline{x} \parallel \underline{u}} = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{m}{2} - 1\right)(k + m - 2)}{2^{k+1}\Gamma\left(k + \frac{m}{2}\right)} C_k^{\frac{m}{2}-1}(1)(|\underline{x}|\, |\underline{u}|)^k,$$

which is a confluent hypergeometric function.

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